Lattice QCD

Continuum Euclidean action:

$$S = \int d^4x \, \underbrace{\frac{1}{2g^2} \operatorname{tr} F^2}_{\downarrow} + \sum_f \bar{q}_f \underbrace{(\not{\partial} + m_f)}_{\downarrow} q_f$$

Wilson /plaquette action

Wilson-Dirac operator coupled to gauge field i.e. replace $\partial \to D$

Lattice QCD action:

$$S_{QCD} = \frac{1}{g^2} \sum_{x\mu\nu} \operatorname{Re} \operatorname{tr} \left\{ 1 - W\left(\boxed{}_{x\mu}^{\nu}\right) \right\} + \sum_f \sum_x a^4 \bar{q}_f \mathcal{D}_f q_f$$

$$\mathcal{D}_f = \not{D}^s + m_f - \frac{ar}{2} \sum_\mu D_\mu^{bw} D_\mu^{fw}$$

$$D_\mu^{fw} q(x) = \frac{1}{a} \left\{ U_\mu(x) q(x + a e_\mu) - q(x) \right\}$$

$$D_\mu^{bw} q(x) = \frac{1}{a} \left\{ q(x) - U_\mu(x - a e_\mu) q(x - a e_\mu) \right\}$$

$$D_\mu^{s} = \frac{1}{2} \left\{ D_\mu^{fw} + D_\mu^{bw} \right\}$$

Path integral measure: $[dU][d\bar{q} dq] = \left[\prod_{x\mu} dU_\mu(x) \right] \left[\prod_{xf\alpha i} d\bar{q}_{f\alpha i}(x) \, dq_{f\alpha i}(x) \right]$

Partition function $Z = \int [dU][d\bar{q} dq] \, e^{-S_{QCD}} = \int [dU] \, e^{-S_{YM}(U)} \prod_f \det\{a^4 \mathcal{D}^f(U)\}$

Lattice QCD in practice

(I) Expectation value of observables $A(U)$ which do not depend on the fermion fields.

E.g. average plaquette $\quad A(U) = \frac{1}{N^4} \sum_{x,\mu\nu} W(\square_{x}^{\mu\nu})$

Wilson loop (useful to extract the static potential) $\quad A(U) = W\left(\square_{x}^{R,T}\right)$

$$\langle A(U) \rangle = \frac{1}{Z} \int [dU]\,[d\bar\psi d\psi]\, e^{-S_{QCD}}\, A(U) \underset{\substack{\uparrow \\ \text{integrate} \\ \text{fermions}}}{=} \frac{1}{Z} \int [dU]\, \underbrace{e^{-S_{YM}(U)}}_{\substack{\text{positive} \\ \text{(since we are} \\ \text{in the Euclidean)}}} \underbrace{\left[\prod_f \det a^4 D^f(U) \right]}_{\substack{\circ\ \text{real in general} \\ \circ\ \text{positive if only} \\ \text{degenerate up \& down}}} A(U)$$

⊛ The Wilson-Dirac operator satisfies γ_5-hermiticity, i.e.

$$\gamma_5 D \gamma_5 = D^\dagger \quad \Rightarrow \quad \det D = \det(\gamma_5 D \gamma_5) = \det(D^\dagger) = (\det D)^*$$
$$\Rightarrow \quad \det D \text{ is real}$$

⊛ Simplified setup: only up & down quarks with $m_u = m_d$

Then $D^u = D^d \equiv D$, $\quad \prod_f \det D^f = (\det D)^2 \geq 0$

(I) Expectation value of observables $A(U)$ which do not depend on the fermion fields.

Simplified setup: two-flavour QCD (only up & down) with $m_u = m_d \equiv m$. Then

$$\langle A(U) \rangle = \int [dU] \, p(U) \, A(U) \qquad \circledast$$

where $p(U) = \frac{1}{Z} e^{-S_{YM}(U)} \left[\det a^4 D(U)\right]^2$ is a probability distribution, i.e.

- $p(U) \geq 0$
- $\int [dU] \, p(U) = 1$

The integral $\circledast$ is calculated with Monte-Carlo method. You design a stochastic process (Markov chain) which produces a sequence of gauge configurations $(U_n)_{n=1,2,3,\ldots}$ distributed according to the probability distribution $p(U)$.

Then:

$$\langle A(U) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} A(U_n)$$

N: length of the Markov chain

(II) Expectation value of observables containing quarks.

E.g. $\bar{u}\gamma_5 d(x)$ is the interpolating operator for the charged pion. The corresponding 2pt function is

$\langle \bar{u}\gamma_5 d(x)\ \bar{d}\gamma_5 u(0) \rangle$

$$= \frac{1}{Z}\int [dU][d\bar{u}\, du][d\bar{d}\, dd]\ e^{-S_{YM}(U) - \sum_x a^4 \bar{u} D u - \sum_x a^4 \bar{d} D d}\ \bar{u}\gamma_5 d(x)\, \bar{d}\gamma_5 u(0)$$

(unfortunate notation...)

you need to exchange u & $\bar{u}$ $\Rightarrow$ (-1) sign!

The fermion integral is Gaussian, and it can always be calculated analytic. It always produce Wick contractions of fermion fields. Each Wick contraction corresponds to D^{-1}:

$$\underbrace{\psi(x)\,\bar{\psi}(y)} \rightarrow D^{-1}(x,y)$$

$$= -\frac{1}{Z}\int [dU]\ e^{-S_{YM}}\left[\det a^4 D(U)\right]^2 \mathrm{tr}_{\substack{\text{spin}\\ \text{color}}}\left\{ D^{-1}(U)(x,0)\,\gamma_5\, D^{-1}(U)(0,x)\,\gamma_5 \right\}$$

$$= -\left\langle \mathrm{tr}_{\substack{\text{spin}\\ \text{color}}}\left\{ D^{-1}(U)(x,0)\,\gamma_5\, D^{-1}(U)(0,x)\,\gamma_5 \right\}\right\rangle$$

(II) Expectation value of observables containing quarks.

E.g. $\bar{u}\gamma_5 d(x)$ is the interpolating operator for the charged pion.
The corresponding 2pt function is

$$\langle \bar{u}\gamma_5 d(x)\, \bar{d}\gamma_5 u(0)\rangle = \begin{bmatrix}\text{integrate quarks}\\ \text{analytically}\end{bmatrix}$$

$$= -\left\langle \mathrm{tr}_{\substack{spin\\color}} \left\{ D^{-1}(U)(x,0)\,\gamma_5\, D^{-1}(U)(0,x)\,\gamma_5 \right\} \right\rangle$$

$$= \int [dU]\, p(U)\, A(U)$$

expectation value of something that depends only on U
→ back to case (I)

with $A(U) = -\mathrm{tr}_{\substack{spin\\color}} \left\{ D^{-1}(U)(x,0)\,\gamma_5\, D^{-1}(U)(0,x)\,\gamma_5 \right\}$

Again, this integral is calculated with Monte-Carlo methods

$$\langle \bar{u}\gamma_5 d(x)\, \bar{d}\gamma_5 u(0)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} A(U_n)$$

Continuum limit in QCD

Simplifying assumption: only up & down (two-flavour QCD) with $m_u = m_d \equiv m$

$$\mathrm{Obs}\,(a, g(a), \hat{m}(a))$$

$\hat{m} = m/a$

- Dim. analysis $[\mathrm{Obs}] = \mathrm{Mass}^{d_{\mathrm{obs}}}$
- lattice predicts only dimensionless

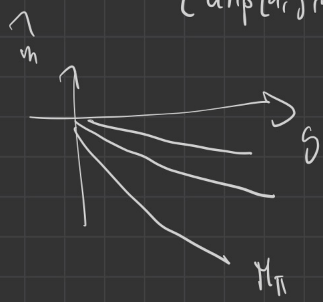
you calculate this

$$\mathrm{Obs}(a, g, \hat{m}) = a^{-d}\,\hat{O}(g, \hat{m})$$

$g(a)$ $\hat{m}(a)$

two conditions e.g.
$$\begin{cases} \dfrac{M_\pi}{M_p}(g,\hat{m}) = \dfrac{\hat{M}_\pi}{\hat{M}_p}(g,\hat{m}) \\ a M_p(a, g, \hat{m}) = M_p^{\mathrm{phys}} \end{cases}$$

From YM:



<u>Statement 3</u>. A function $g(a)$ exists such that

$\lim_{a\to 0} M_X(a, g(a)) = M_X^{\mathrm{cont}}$ is finite and $\neq 0$ for all X

$\lim_{a\to 0} \sigma(a, g(a)) = \sigma^{\mathrm{cont}}$ " " " " "

$\lim_{a\to 0} F(R\,|\,a, g(a)) = F^{\mathrm{cont}}(R)$ " " " " " for every $R \neq 0$

(a) The same function $g(a)$ works for all observables above

(b) The function $g(a)$ is not unique.

(c) However $\lim_{a\to 0} C_X(x\,|\,a, g(a)) = +\infty$ [operators require extra renormalization]